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# Magnetohydrodynamic flow of a Sisko fluid in annular pipe: A numerical study

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SUMMARY

This paper presents a numerical study for the unsteady flow of a magnetohydrodynamic (MHD) Sisko fluid in annular pipe. The fluid is assumed to be electrically conducting in the presence of a uniform magnetic field. Based on the constitutive relationship of a Sisko fluid, the non-linear equation governing the flow is first modelled and then numerically solved. The effects of the various parameters especially the power index $n$, the material parameter of the non-Newtonian fluid $b$ and the magnetic parameter $B$ on the flow characteristics are explored numerically and presented through several graphs. Moreover, the shear-thinning and shear-thickening characteristics of the non-Newtonian Sisko fluid are investigated and a comparison is also made with the Newtonian fluid. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: MHD flow; Sisko fluid; annular pipe; numerical study; non-Newtonian fluid; incompressible flow; laminar flow; partial differential equations; finite difference methods

## 1. INTRODUCTION

The majority of the literature deals with the flows of viscous fluid described by means of the classical Newtonian model. However, there are many rheological complex fluids such as polymeric liquids, drilling mud, paints, lubricating oils, biological fluids and so forth for which the classical Navier–Stokes theory is inadequate. The study of such fluids has gained much interest in recent years because of their numerous industrial and technological applications. Such fluids are often referred to as non-Newtonian fluids. Typical non-Newtonian flow characteristics include shear-thinning, shear-thickening, viscoelasticity, viscoplasticity and so forth. For the flows of non-Newtonian fluids there is not a single model that describes all of their properties as is done for the Newtonian fluid. Over the last few years, various investigations have been made to study the various flow problems of non-Newtonian fluids [1–10]. A good number of fluid rheologies are already in existence, particularly for most of those fluids used as lubricants and possessing non-Newtonian characteristics. The flows

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of such fluids can be analyzed with the help of a power-law model. However, now in addition to viscosity, another parameter, namely the power-law index (or exponent) is used to characterize fluids. These are vital as well. Some recent studies regarding power-law fluids are presented in References [11–16]. Additional pertinent literature can be recovered in the bibliography reported in these studies.

The study of an electrically conducting fluid flow under a transversely applied magnetic field has become the basis of many scientific and engineering applications. There has been great interest in the study of magnetohydrodynamic (MHD) flow due to the effect of magnetic fields on the boundary layer control and on the performance of many systems using electrically conducting fluids. Specifically, the interest in MHD fluid flow stems because of its applications in many devices such as MHD power generators, accelerators, centrifugal separation of matter from fluid, fluid droplet sprays, purification of crude oil, petroleum industry, polymer technology and so forth. The flows of non-Newtonian fluids in the presence of a magnetic field have been studied earlier by several authors [17–21] considering mostly steady-state analysis. The purpose of the present paper is to study time-dependent flow characteristics of non-Newtonian Sisko fluid in the presence of a magnetic field. The reason for considering a Sisko fluid is that the Sisko fluid can demonstrate many typical characteristics of Newtonian and non-Newtonian fluids by choosing different material parameters.

In view of the above motivations, in the present paper, we study the MHD flow of a Sisko fluid in annular pipe. The fluid is assumed to be electrically conducting in the presence of a uniform magnetic field. Further, there is no external electric field imposed on the fluid and the magnetic Reynolds number is also assumed to be very small. In this paper, numerical computations are performed using a high-order finite difference scheme based on the summation by parts (SBP) operators [22]. In addition, the treatment of boundary conditions is done using the simultaneous approximation term (SAT) method [23] such that these SBP operators with SAT boundary treatment lead to a stable scheme. The remainder of the paper is organized as follows. In the following section, the constitutive equations and equations of motion for Sisko fluid are given. Section 3 contains the description of the problem. Section 4 is devoted to the numerical procedure. Numerical results and discussion are presented in Section 5. The paper ends with a brief summary.

## 2. GOVERNING EQUATIONS

The field equations governing the transient flow of an incompressible MHD fluid are

$$\operatorname{div}\mathbf{V}=0 \tag{1}$$

$$\rho\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=\operatorname{div}\mathbf{T}+\mathbf{J}\times\mathbf{B} \tag{2}$$

where $\mathbf{V}$ is the velocity field, $\rho$ the uniform density of the fluid, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{J}$ the current density, $\mathbf{B}$ the total magnetic field and $\mathrm{d}/\mathrm{d}t$ the material time derivative.

The constitutive equations for a Sisko fluid are [11–14]

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S},\quad \mathbf{S}=[a+b|\sqrt{\Pi}|^{n-1}]\mathbf{A} \tag{3}$$

with the expressions

$$\mathbf{A}=\mathbf{L}+\mathbf{L}^{\mathsf{T}},\quad \mathbf{L}=\operatorname{grad}\mathbf{V},\quad \Pi=\tfrac{1}{2}\operatorname{tr}(\mathbf{A}^2) \tag{4}$$



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In the above equations, $p$ is the pressure, $\mathbf{I}$ the identity tensor, $\mathbf{S}$ the extra stress tensor, $\mathbf{A}$ the rate of deformation tensor, $\Pi$ the second invariant of the symmetric part of the velocity gradient, $n$, $a$ and $b$ are the material parameters defined differently for different fluids. Note that for $a=0$ the generalized power-law model is recovered and for $b=0$ the Newtonian fluid model can be obtained.

We seek the velocity and the stress fields of the form

$$\mathbf{V}=\mathbf{V}(r,t)=w(r,t)\mathbf{e}_z, \quad \mathbf{S}=\mathbf{S}(r,t) \tag{5}$$

where $\mathbf{e}_z$ is the unit vector in the $z$-direction of the cylindrical polar coordinates system.

A uniform magnetic field $\mathbf{B}_0$ is applied in the transverse direction to the fluid. The magnetic Reynolds number is assumed to be very small so that the induced magnetic field is neglected [18]. Hence, the MHD body force caused by the external magnetic field takes the form

$$\mathbf{J}\times\mathbf{B}=(0,0,-\sigma B_0^2 w) \tag{6}$$

in which $B_0$ is the magnitude of $\mathbf{B}_0$ and $\sigma$ the electrical conductivity of the fluid.

Invoking Equation (5), the constraint of incompressibility is automatically satisfied and from Equation (3) the non-zero component of stress is

$$S_{rz}=\left(a+b\left|\frac{\partial w}{\partial r}\right|^{n-1}\right)\frac{\partial w}{\partial r} \tag{7}$$

Accordingly, the $z$-component of the equation of motion (2) along with Equation (6) gives

$$\rho\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r S_{rz})-\sigma B_0^2 w \tag{8}$$

where $r$ and $\theta$ components of the equation of motion yield that $p$ is independent of $r$ and $\theta$. In addition, $z$ differential of pressure in Equation (8) is constant since the flow is due to the prescribed pressure gradient.

Elimination of $S_{rz}$ between Equations (7) and (8) leads to the following governing equation:

$$\rho\frac{\partial w}{\partial t}=-\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{\partial}{\partial r}\left(\left[a+b\left|\frac{\partial w}{\partial r}\right|^{n-1}\right]\frac{\partial w}{\partial r}\right)+\frac{1}{r}\left(a+b\left|\frac{\partial w}{\partial r}\right|^{n-1}\right)\frac{\partial w}{\partial r}-\sigma B_0^2 w \tag{9}$$

## 3. DESCRIPTION OF THE PROBLEM

Let us suppose that an incompressible electrically conducting Sisko fluid at rest occupies an annular space between two concentric cylinders. At time $t=0^+$, the fluid starts its motion suddenly due to a constant pressure gradient in the $z$-direction, which is taken as the axis of the flow. Its velocity is of the form (5) and the governing equation is (9) while the appropriate initial and boundary conditions are

$$w(r,0)=0, \quad R_0\leqslant r\leqslant R_1 \tag{10}$$

$$w(R_0,t)=w(R_1,t)=0, \quad t>0 \tag{11}$$

where $R_0$ and $R_1$ are the radius of inner and outer cylinders, respectively.



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To non-dimensionalize Equations (9)–(11), we introduce the dimensionless parameters

$$w^* = \frac{w}{U_0}, \quad r^* = \frac{r}{R_0}, \quad z^* = \frac{z}{R_0}, \quad t^* = \frac{at}{\rho R_0^2}, \quad p^* = \frac{p}{(aU_0)/R_0}$$
$$b^* = \frac{b}{a}\left|\frac{U_0}{R_0}\right|^{n-1}, \quad B^2 = \frac{\sigma B_0^2}{(a/R_0^2)}, \quad d = \frac{R_1}{R_0} \tag{12}$$

where $U_0$ is the reference velocity.

The dimensionless problem after dropping asterisks for simplicity takes the following form:

$$\frac{\partial w}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}z} + \frac{\partial}{\partial r}\left(\left[1+b\left|\frac{\partial w}{\partial r}\right|^{n-1}\right]\frac{\partial w}{\partial r}\right) + \frac{1}{r}\left(1+b\left|\frac{\partial w}{\partial r}\right|^{n-1}\right)\frac{\partial w}{\partial r} - B^2 w \tag{13}$$

$$w(r,0) = 0, \quad 1 \leqslant r \leqslant d \tag{14}$$

$$w(1,t) = w(d,t) = 0, \quad t > 0 \tag{15}$$

In the next section we present the numerical technique used to solve the above problem.

## 4. NUMERICAL PROCEDURE

We solve the problem in Equations (13)–(15) using a high-order finite difference scheme in a space that is based on the SBP operators together with the SAT method [22–25] (for SBP–SAT theory, see also references therein). While integration in time is performed using the fourth-order Runge–Kutta method, we also compute steady-state solutions in some cases that are obtained by taking the residual in time with a tolerance of $1 \times 10^{-12}$. All the computations are performed in an

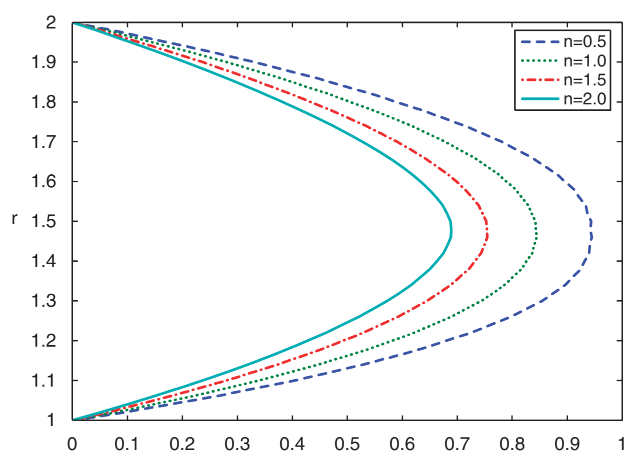


Figure 1. Profiles of the axial velocity $w(r,t)$ for different values of power index $n$. The other parameters chosen are $t=0.5$, $b=0.5$, $B=0$ (hydrodynamic fluid) and $\mathrm{d}p/\mathrm{d}z = -10$.

interval $[1, d]$ with $d=2$ using 100 grid points (equidistant grid) in the computational domain. We find that the resolution of the results is good enough as shown by the figures in the next section.

## 5. NUMERICAL RESULTS AND DISCUSSION

We have studied numerically, in the present paper, the MHD flow of a Sisko fluid in an annular pipe. The highly non-linear problem consisting of differential equation (13) and conditions (14) and (15) is solved numerically using the method described in Section 4. In order to get a clear insight into the physical problem, the velocity profiles $w(r,t)$ have been discussed by assigning the

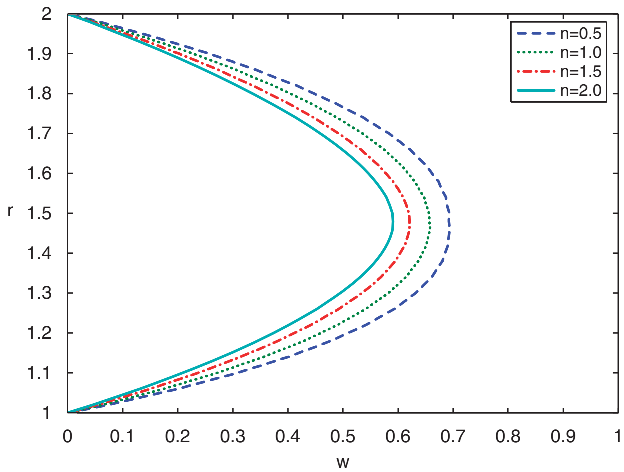


Figure 2. Profiles of the axial velocity $w(r,t)$ for different values of power index $n$. The other parameters chosen are $t=0.5$, $b=0.5$, $B=2$ (hydromagnetic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.

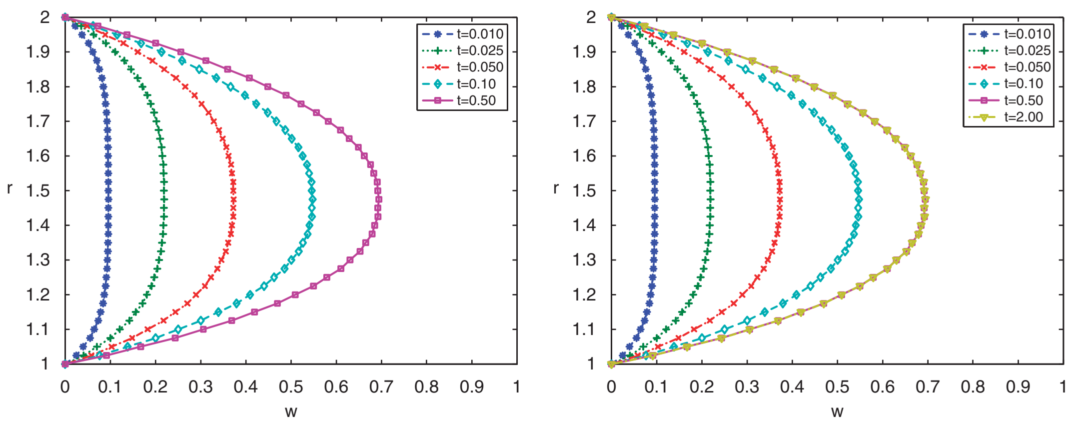


Figure 3. Profiles of the axial velocity $w(r,t)$ for different values of time $t$. The other parameters chosen are $n=0.5$, $b=0.5$, $B=2$ (hydromagnetic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.


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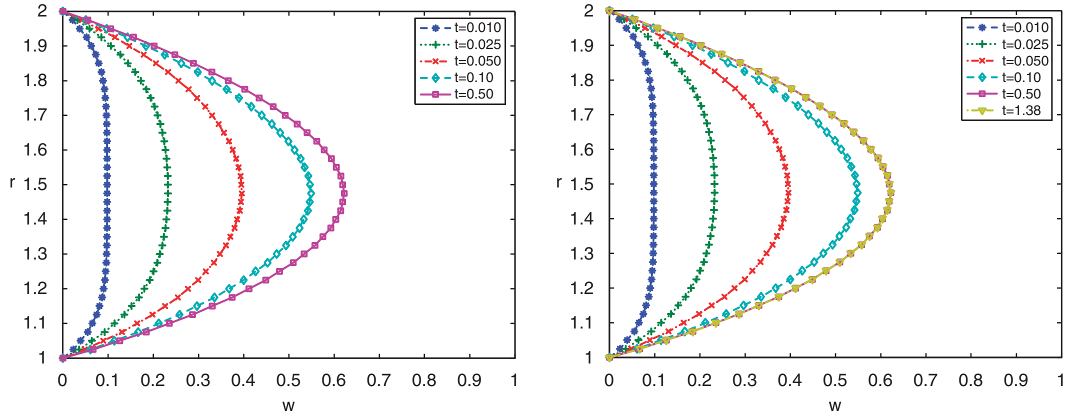


Figure 4. Profiles of the axial velocity $w(r,t)$ for different values of time $t$. The other parameters chosen are $n=1.5$, $b=0.5$, $B=2$ (hydromagnetic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.

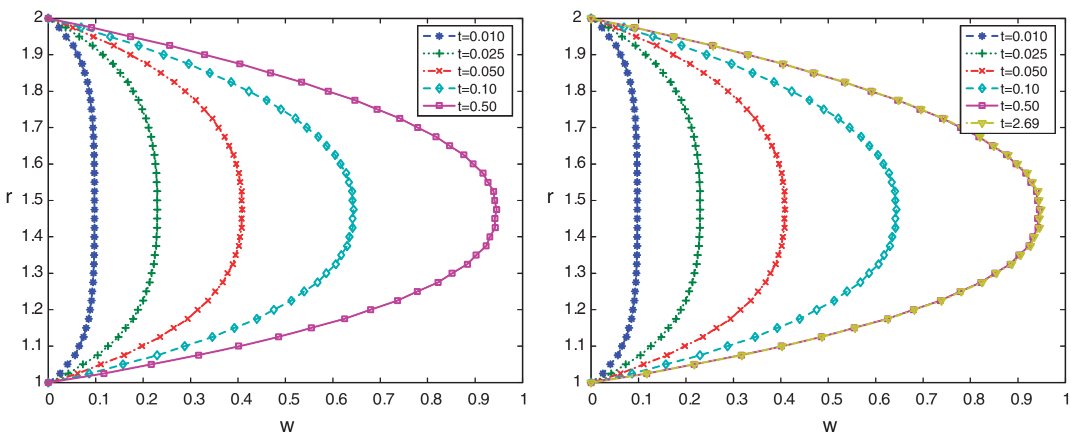


Figure 5. Profiles of the axial velocity $w(r,t)$ for different values of time $t$. The other parameters chosen are $n=0.5$, $b=0.5$, $B=0$ (hydrodynamic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.

numerical values to the non-dimensional parameters encountered in the problem. The numerical results are shown graphically in Figures 1–14. We compare the profiles of velocity for two kinds of fluids: a Newtonian fluid (when $b=0$) and a Sisko fluid (when $b\neq 0$). A comparison is also performed for hydrodynamic and MHD flows. Further, to make an unsteady analysis, by choosing between different parameters, we take fixed time as $t=0.5$. The time is selected in such a way that the flow field is developed.

The effect of the power index $n$ on the velocity is illustrated in Figures 1 and 2 by choosing four different values of $n$ in the absence as well as in the presence of magnetic field $B$, respectively. From these figures, it can be seen that the velocity decreases substantially with the increase of $n$. Thus, the effect of increasing the values of power index $n$ is to reduce the velocity thereby reducing

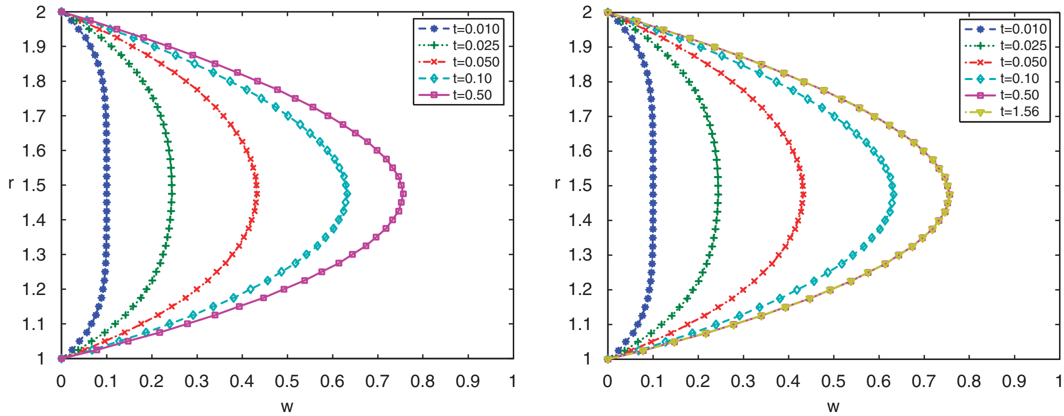


Figure 6. Profiles of the axial velocity $w(r,t)$ for different values of time $t$. The other parameters chosen are $n=1.5$, $b=0.5$, $B=0$ (hydrodynamic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.

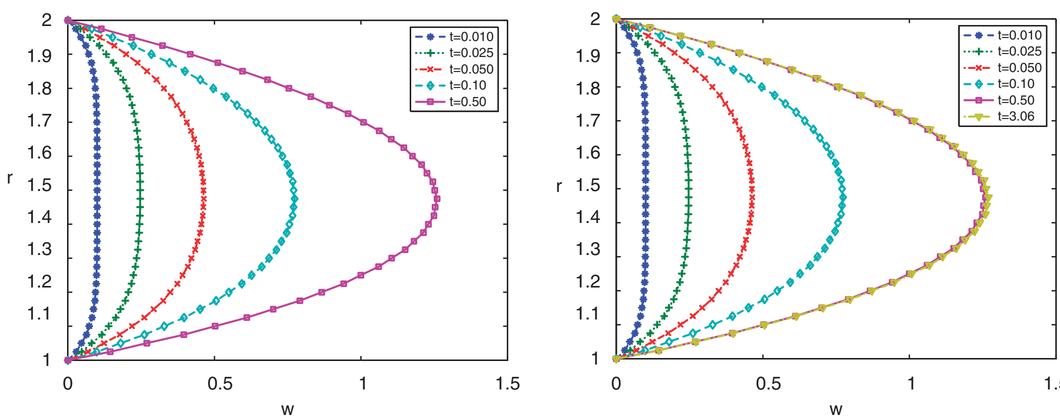


Figure 7. Profiles of the axial velocity $w(r,t)$ for different values of time $t$. The other parameters chosen are $b=0$ (Newtonian fluid), $B=0$ (hydrodynamic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.

the boundary layer thickness. Moreover, these figures also indicate that the effect of power index $n$ on hydrodynamic flow is more significant than hydromagnetic flow.

Generally, the unsteady flows of non-Newtonian fluids are important for those who need to eliminate transients from their rheological measurement. Consequently, it is necessary to know the approximate required time to reach the steady state. For this, the velocity profile for various values of time is shown in Figures 3–8. It is noted that at small time, the difference between the velocity profiles is larger and this difference rapidly decreases for larger values of time. From Figures 3 and 4, the required time to get steady state is $t=2$ for $n=0.5$ and it is $t=1.38$ for $n=1.5$. Figures 5 and 6 indicate that the steady state is reached at $t=2.69$ for $n=0.5$ and $t=1.56$ for $n=1.5$. This shows that the required time to reach steady state for hydromagnetic flow (when $B=2$) is smaller than that for hydrodynamic flow (when $B=0$). Further, as it results from Figures 7



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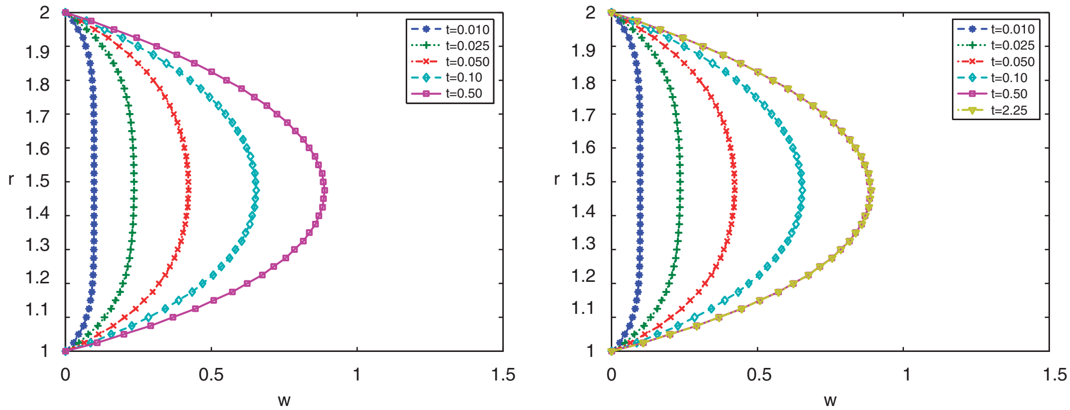


Figure 8. Profiles of the axial velocity $w(r,t)$ for different values of time $t$. The other parameters chosen are $b=0$ (Newtonian fluid), $B=2$ (hydromagnetic fluid) and $\mathrm{d}p/\mathrm{d}z=-10$.

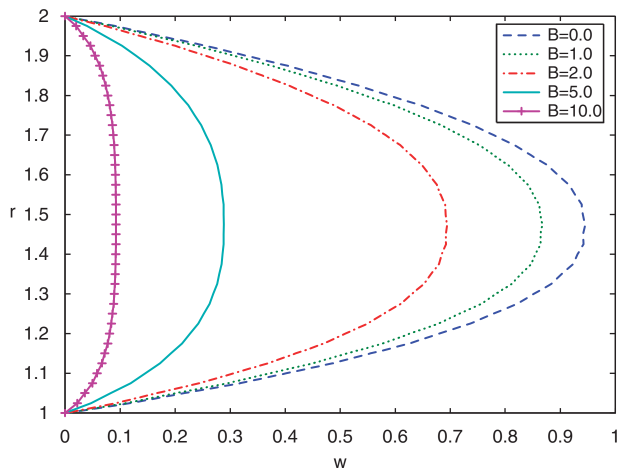


Figure 9. Profiles of the axial velocity $w(r,t)$ for different values of magnetic parameter $B$. The other parameters chosen are $n=0.5$, $b=0.5$, $t=0.5$ and $\mathrm{d}p/\mathrm{d}z=-10$.

and 8, the required time to get the steady state for Newtonian fluid (when $b=0$) is $t=3.06$ for hydrodynamic flow (when $B=0$) and it is $t=2.25$ for hydromagnetic flow (when $B=2$). Hence, it can be predicted that the required time to get the steady state for Sisko fluid is smaller than that of Newtonian fluid.

Figures 9–11 are plotted for the variation of magnetic parameter $B$ with three different power indexes $n$. From these figures, it is noted that the velocity decreases with increase in the magnetic parameter $B$ in all cases. This is due to the fact that the introduction of transverse magnetic field has a tendency to develop a drag that tends to resist the flow.



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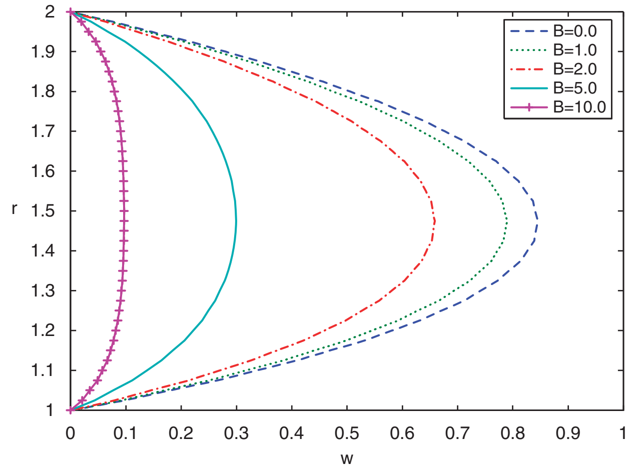


Figure 10. Profiles of the axial velocity $w(r,t)$ for different values of magnetic parameter $B$. The other parameters chosen are $n=1.0$, $b=0.5$, $t=0.5$ and $\mathrm{d}p/\mathrm{d}z=-10$.

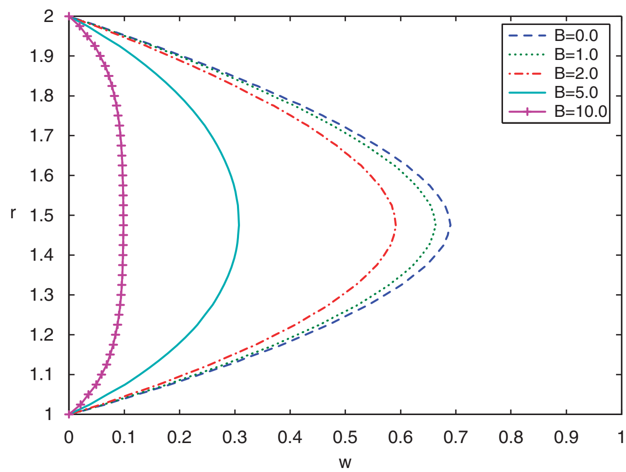


Figure 11. Profiles of the axial velocity $w(r,t)$ for different values of magnetic parameter $B$. The other parameters chosen are $n=2.0$, $b=0.5$, $t=0.5$ and $\mathrm{d}p/\mathrm{d}z=-10$.

To see the effect of the rheology of the fluid, Figures 12–14 with power index values ($n=0.5, 1, 2$) are prepared. These figures also display a comparison between a Newtonian fluid (when $b=0$) and a Sisko fluid (when $b\neq 0$). From these graphs, we observe that the increasing material parameter $b$ reduces the velocity substantially in all cases. This reduction in the velocity is much for $n=2$ when compared with that for $n=0.5$ and $n=1$, which indicate a shear-thickening phenomenon of the examined non-Newtonian fluid. Further, from these figures, it is clear that the velocity profile for a



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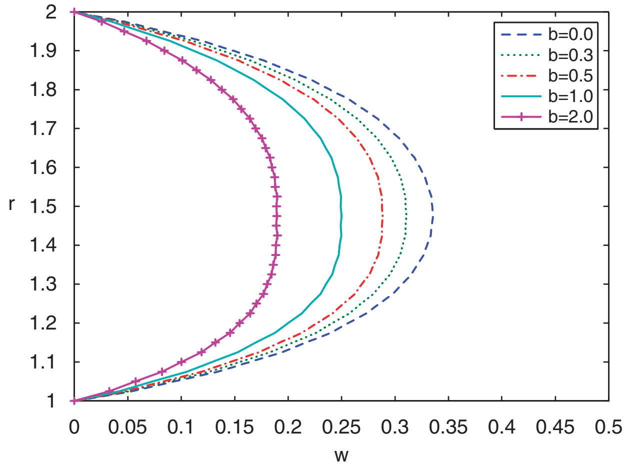


Figure 12. Profiles of the axial velocity $w(r,t)$ for different values of material parameter $b$. The other parameters chosen are $n=0.5$, $B=5$, $t=0.5$ and $\mathrm{d}p/\mathrm{d}z=-10$.

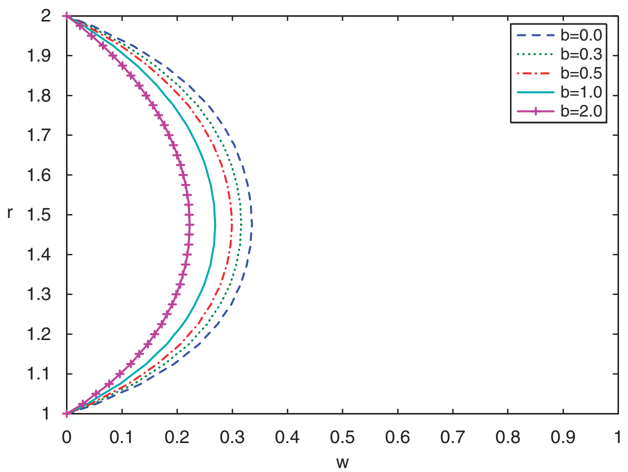


Figure 13. Profiles of the axial velocity $w(r,t)$ for different values of material parameter $b$. The other parameters chosen are $n=1.0$, $B=5$ (hydromagnetic fluid), $t=0.5$ and $\mathrm{d}p/\mathrm{d}z=-10$.

Newtonian fluid is much larger when compared with the Sisko fluid. This indicates that rheology of the fluid has significant effects on the flow.

## 6. BRIEF SUMMARY

In the present study, we have investigated magnetohydrodynamic (MHD) flow of a Sisko fluid in an annular pipe. The system is stressed by a uniform transverse magnetic field. The velocity

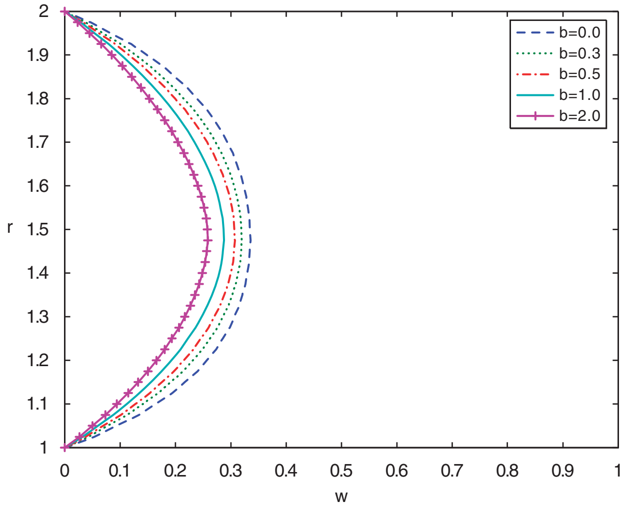


Figure 14. Profiles of the axial velocity $w(r,t)$ for different values of material parameter $b$. The other parameters chosen are $n=2.0$, $B=5$ (hydromagnetic fluid), $t=0.5$ and $\mathrm{d}p/\mathrm{d}z=-10$.

profiles are obtained using the fourth-order Runge–Kutta method for various values of the physical parameters. A comparison between the Newtonian and non-Newtonian Sisko fluids is also made. The findings from the work are summarized as follows:

- It is observed that the power-law index has the effect to decrease the velocity profile and reduce the boundary layer thickness.
- It is noted that the velocity profiles for a Newtonian fluid are much greater in magnitude than those for a Sisko fluid.
- It is observed that the velocity decreases monotonically by increasing the magnetic parameter $B$ for $n=0.5$, 1.5 and 2.
- The effects of the material parameter $b$ on the velocity profile are similar to those of magnetic parameter $B$.
- It is seen that the required time to reach steady state for hydromagnetic flow is smaller than that of hydrodynamic flow for all $n$.
- This study shows that the required time to get the steady state for Sisko fluid is smaller than that of Newtonian fluid for $n=0.5$ and 1.5.

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